

MATH 245 S23, Exam 1 Solutions

- Carefully define the following terms: odd, predicate.
Let n be an integer. We call n odd if there exists some integer m satisfying $n = 2m + 1$. A predicate is a collection of propositions, indexed by one or more free variables, each drawn from its domain.
- Carefully state the following theorems: Division Algorithm Theorem, Conditional Interpretation Theorem.
The Division Algorithm Theorem says: For any integers a, b with $b \geq 1$, there are unique integers q, r satisfying $a = bq + r$ and $0 \leq r < b$. The Conditional Interpretation theorem states: for any propositions p, q , we have $p \rightarrow q \equiv q \vee \neg p$.
- Without using truth tables, prove: For all propositions p, q , we have $(p \rightarrow q) \wedge (p \rightarrow \neg q) \equiv \neg p$.

METHOD 1: Use Theorem 3.2, i.e. prove \vdash and \dashv separately then combine to get \equiv .

(a) Suppose first that $(p \rightarrow q) \wedge (p \rightarrow \neg q)$ is true. By simplification twice, we get $p \rightarrow q$ and $p \rightarrow \neg q$. Two cases: Case p is false. Then $\neg p$ is true, and we are done. Case p is true. Then by modus ponens twice, we get both q and $\neg q$, which is impossible. Hence $\neg p$ is true.

(b) Suppose now that $\neg p$ is true. By addition $q \vee \neg p$ is true. By conditional interpretation $p \rightarrow q$ is true. We do this again: by addition $(\neg q) \vee \neg p$ is true, and by conditional interpretation $p \rightarrow \neg q$ is true. By conjunction $(p \rightarrow q) \wedge (p \rightarrow \neg q)$ is true.

METHOD 2: Chain of logical equivalences. Each step must be justified for full credit.

$(p \rightarrow q) \wedge (p \rightarrow \neg q) \equiv (q \vee \neg p) \wedge ((\neg q) \vee \neg p) \equiv ((\neg p) \vee q) \wedge ((\neg p) \vee \neg q) \equiv (\neg p) \vee (q \wedge \neg q) \equiv (\neg p) \vee F$
The first \equiv is from conditional interpretation twice. The second \equiv is from the commutativity theorem twice. The third \equiv is from the distributivity theorem (backwards). The fourth \equiv is from a theorem in the book (Thm 2.7f). Lastly, $(\neg p) \vee F \vdash \neg p$ by disjunctive syllogism, and $\neg p \vdash (\neg p) \vee F$ by addition.

METHOD 3: Working through four cases in detail using words (e.g. Suppose p is T and q is F , then $p \rightarrow q$ is F , etc.) is not *technically* using a truth table, so if all the details are present this rather long solution would also earn full points. My mistake – instead of “without truth tables” I should have written “using semantic theorems”.

- Use a truth table to help prove: For all propositions p, q , we have $(p \rightarrow q) \wedge (p \rightarrow \neg q) \equiv \neg p$. In this truth table, the third and seventh columns agree, which proves the desired conclusion.

p	q	$\neg p$	$\neg q$	$p \rightarrow q$	$p \rightarrow \neg q$	$(p \rightarrow q) \wedge (p \rightarrow \neg q)$
T	T	F	F	T	F	F
T	F	F	T	F	T	F
F	T	T	F	T	T	T
F	F	T	T	T	T	T

- Let $a, b, c \in \mathbb{Z}$. Suppose that $a \leq b$ and $0 \leq c$. Prove that $ac \leq bc$.
Because $a \leq b$, we must have $b - a \in \mathbb{N}_0$. Because $0 \leq c$, we must have $c = c - 0 \in \mathbb{N}_0$. The product of two whole numbers is a whole number, so $(b - a)c \in \mathbb{N}_0$. Multiplying out, we get $bc - ac \in \mathbb{N}_0$. Hence, $ac \leq bc$.
- For arbitrary $n \in \mathbb{N}_0$, calculate and simplify $\frac{(n+2)!}{(n+4)!}$

Using the definition of factorial twice, we have $\frac{(n+2)!}{(n+4)!} = \frac{(n+2)!}{(n+3)!(n+4)} = \frac{(n+2)!}{(n+2)!(n+3)(n+4)} = \frac{1}{(n+3)(n+4)}$. If desired, this can be expanded as $\frac{1}{n^2+7n+12}$.

7. Simplify the following expression as much as possible, where only basic propositions are negated. Be sure to justify every step. $\neg(p \vee \neg(p \vee \neg(q \vee r)))$

Applying De Morgan's law to the outermost \neg , we get $(\neg p) \wedge \neg\neg(p \vee \neg(q \vee r))$.

Applying Double Negation, we get $(\neg p) \wedge (p \vee \neg(q \vee r))$.

Applying De Morgan's law to the innermost \neg , we get $(\neg p) \wedge (p \vee ((\neg q) \wedge (\neg r)))$.

NOTE: Applying Disjunctive Syllogism would give $(\neg q) \wedge (\neg r)$; however, this is not logically equivalent to the previous step, so is not a correct simplification. $(\neg p) \wedge (\neg q) \wedge (\neg r)$ would be correct, but is tricky to justify.

8. Prove or disprove: $\forall x \in \mathbb{Z}, |4x - 9| > 1$.

The statement is false, and requires a counterexample. Take $x^* = 2$. We have $|4x^* - 9| = |8 - 9| = |-1| = 1$, and $1 \not> 1$. Hence for $x^* = 2$, we have $|4x^* - 9| \not> 1$.

9. Prove or disprove: $\forall x \in \mathbb{Z}, x \leq 1 \rightarrow |4x - 9| > 1$.

The statement is true. We must begin by letting $x \in \mathbb{Z}$ be arbitrary.

SOLUTION 1: Direct proof. Assume $x \leq 1$. Hence $4x \leq 4 \cdot 1 = 4$, and $4x - 9 \leq 4 - 9 = -5$. In particular, $4x - 9 < 0$, so $|4x - 9| = -(4x - 9) \geq -(-5) = 5 > 1$. Putting it all together, $|4x - 9| > 1$.

SOLUTION 2: Contrapositive proof. Assume that $|4x - 9| \leq 1$, i.e. $-1 \leq 4x - 9 \leq 1$. We add 9 to get $8 \leq 4x \leq 10$, then divide by 4 to get $2 \leq x \leq 2.5$. Since $x \in \mathbb{Z}$, we conclude that $x = 2$. Now $2 > 1$, i.e. $x > 1$.

SOLUTION 3: Proof by cases. Case $x > 1$: The implication is true vacuously.

Case $x \leq 1$: Proceed as in Solution 1. $4x \leq 4 \cdot 1 = 4$, then $4x - 9 \leq 4 - 9 = -5$. In particular, $4x - 9 < 0$, so $|4x - 9| = -(4x - 9) \geq -(-5) = 5 > 1$. Putting it all together, $|4x - 9| > 1$. Now the implication is true trivially.

10. Prove or disprove: $\forall x \in \mathbb{Z}, |4x - 9| > 1 \rightarrow x \leq 1$.

The statement is false, and requires a counterexample. Hence we need an explicit, specific $x^* \in \mathbb{Z}$ where $\neg(p(x^*) \rightarrow q(x^*))$ holds, i.e. $p(x^*) \wedge \neg q(x^*)$.

Take $x^* = 10$. We have $|4x^* - 9| = |40 - 9| = |31| = 31 > 1$, but also $x^* = 10 \not\leq 1$.